The Optical Ring Resonator

By W. W. RIGROD

(Manuscript received February 1, 1965)

Expressions are derived for the stability parameter, spot size, and wavefront curvature of a Gaussian beam in a ring resonator containing up to four spherical mirrors unequally spaced. Higher-order transverse modes and aperture effects are not considered. Two methods of analysis are used: (1) replacement of the mirrors by an infinite sequence of equally-spaced identical thick lenses, and (2) transformation of the beam into itself after one circuit of the ring, by means of a ray matrix representation of the equivalent thin lenses. The procedure can readily be extended to ring resonators with any number of spherical mirrors.

I. INTRODUCTION

In the ring laser, a light beam is directed about a closed loop, typically by three or four mirrors, and regeneratively amplified at frequencies for which the circuit path equals an integral number of wavelengths.[1-3] The only available analysis of a ring resonator with more than one spherical mirror appears to be that of Clark,[4] who used a ray-optical approach to derive the stability conditions for a ring with mirrors of two different curvatures and spacings. However, the means for a complete analysis of any arbitrary ring resonator are contained implicitly in the optical-mode theory developed in recent years in connection with two-mirror resonators.[5-8] The purpose of this note is to trace the connection between this theory and that of ring resonators in two different ways, and to derive the formulae defining the Gaussian (fundamental mode) beam in an arbitrary four-mirror resonator. A third method has been proposed recently by Collins in general form,[9,10] but will not be employed here because of its greater complexity.

In a two-mirror resonator, the wavefront curvatures of the light beam coincide with those of the mirrors, since the beams are reflected back on themselves. This is not the case in ring resonators, in which the beam is reflected obliquely. The boundary condition of the latter is merely that

the beam reproduce itself after each circuit, following its passage through a series of focusing elements or equivalent lenses.

As noted by Boyd and Kogelnik,[7] the stability conditions and beam size in spherical mirror resonators are the same as in an equivalent sequence of lenses. Thus the problem consists of applying the traveling-wave boundary condition to the appropriate equivalent-lens system. Owing to the astigmatism of concave mirrors in oblique reflection, they must be replaced by two different sequences of lenses, and each analyzed separately. For example, in the quadrilateral resonator of Fig. 1(a), the equivalent lens sequence for the clockwise traveling wave is shown in Fig. 1(b), where the focal length of the ith lens is given by[11]

$$f_{xi} = \tfrac{1}{2} b_i \cos (\varphi_i/2) \tag{1a}$$

in the plane of the ring, and

$$f_{yi} = b_i/2 \cos (\varphi_i/2) \tag{1b}$$

in the plane normal to the ring, for a mirror with radius of curvature b_i,

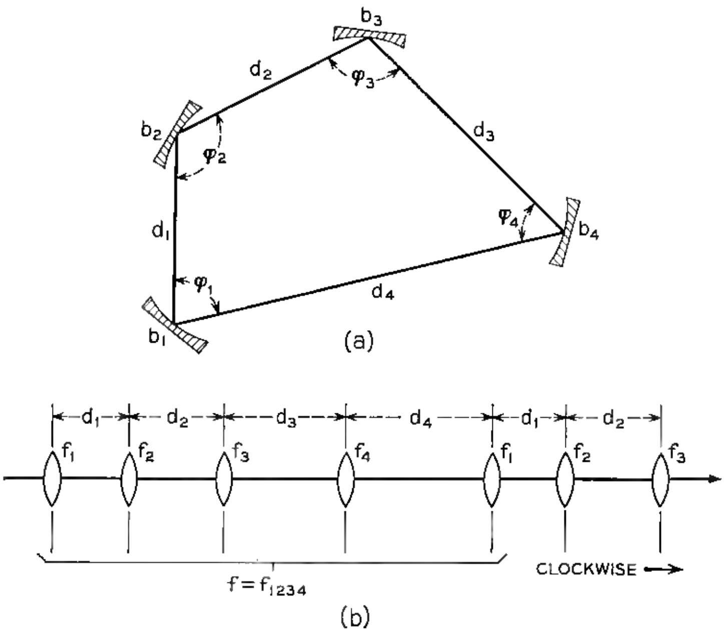

Fig. 1 — (a) Ring resonator with four spherical mirrors, unequally spaced. (b) Equivalent sequence of thin lenses for clockwise wave, with focal lengths given by equations (1a) and (1b) for tangential and sagittal planes, respectively.

subtending an included angle φ_i. The beam, consequently, is usually elliptical in cross section.

The circulating beam in a ring laser can be made circular in cross section, if need be, by the use of astigmatic mirrors or lenses, such that the effective curvatures of the ith mirror, b_{xi} and b_{yi}, in and normal to the plane of the ring, respectively, are related through

$$b_{yi} = b_{xi} \cos^2 (\varphi_i/2). \tag{2}$$

When the laser beam is plane-polarized, it is possible to design such correcting lenses for insertion at the Brewster angle to the optic axis, to minimize transmission losses. Alternatively, the elliptic output beam of a ring laser can be transformed into one of circular cross section by means of a single astigmatic element outside of the ring, placed where the spot is circular.

II. EQUIVALENT SEQUENCE OF THICK LENSES

The first method of analyzing the iterated sequence of four thin lenses shown in Fig. 1(b) is to replace them by a sequence of identical thick lenses f, whose principal planes are separated by a constant distance L. The stability condition for this system,[12]

$$0 < \frac{L}{f} < 4 \tag{3}$$

is then expressed in terms of the focal lengths and spacings of the equivalent thin lenses, and the beam description (radius, wavefront curvature, etc.) obtained from the known relations for an equivalent two-mirror resonator.[5-8]

The beam waist w_0 is located at a distance $L/2$ from the principal planes of each thick lens, and is given by

$$\frac{2\pi w_0^2}{\lambda} = L\left(\frac{4f}{L} - 1\right)^{\frac{1}{2}} \tag{4}$$

where λ is the wavelength. At a distance z from the waist in real space (when there are no intervening lenses), the beam radius is given by

$$w = w_0\left[1 + \left(\frac{z\lambda}{\pi w_0^2}\right)^2\right]^{\frac{1}{2}} \tag{5}$$

and the wavefront curvature R by

$$R = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right] \tag{6}$$

where R has the same sign as z, positive when the center of curvature is to the left of the surface, i.e., when the surface is to the right of the beam waist.

The detailed procedure is illustrated in Fig. 2, in which the four thin lenses are taken in the order f_1, $d_1 \cdots f_4$, d_4. First f_1 and f_2 are combined to form a thick lens f_{12} with principal planes located at distances h_1 and h_2 from the two lenses; then f_{12} and f_3 are combined to form the thick lens f_{123} with its principal planes located at distances h_1' and h_2' from the principal planes of the component lenses; and similarly for the combination of f_{123} and f_4 to form f_{1234}. The principal planes of f_{1234} are located at distances H_1 and H_2, respectively, from each of the thin lenses f_1 and f_4 as shown.

The value of $f = f_{1234}$, as well as of L_4, depends on the way in which the thin lenses are grouped, i.e., $f_{1234} \neq f_{2341}$. Thus, whereas L/f is invariant for the group of iterated thin lenses, there can be as many different beam waists as there are lenses. For the group shown in Fig. 2, the waist defined by L_4 in (4) above is located at a distance S_4 to the right of lens f_4 (i.e., measured clockwise from mirror b_4 in the ring resonator). Given the location and radius of the beam waist, the beam size and wavefront curvature at any distance z from the waist can be computed from the foregoing relations.

The expressions for the significant parameters of the equivalent thick-

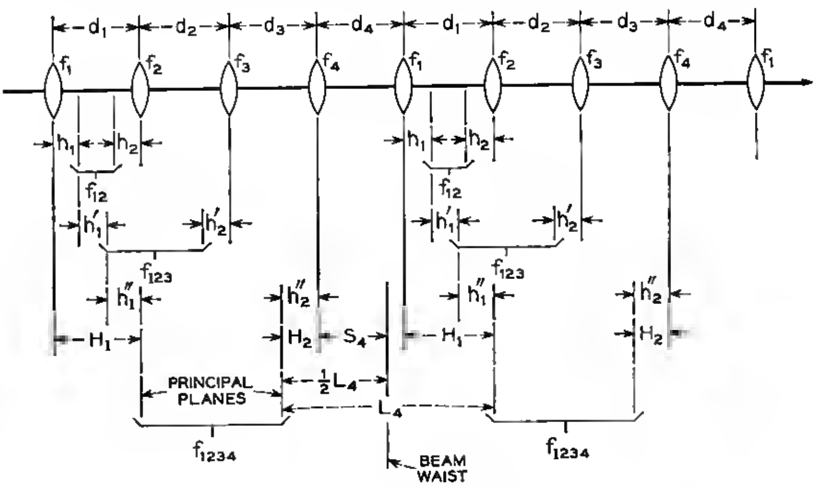

Fig. 2 — Reduction of iterated four-lens system to sequence of identical thick lenses ($f = f_{1234}$).

lens sequence of Fig. 2 are listed below. The remaining three beam waists in this system of four lenses can be found either by imaging the known waist about its nearest lens,[8,13] or simply by permuting the indices for the thin lenses and their spacings.

$$\begin{aligned}\frac{1}{f_{1234}} = {} & \left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}\right) - \frac{d_1}{f_1}\left(\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}\right) - \frac{d_3}{f_4}\left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}\right) \\ & - d_2\left(\frac{1}{f_1}+\frac{1}{f_2}\right)\left(\frac{1}{f_3}+\frac{1}{f_4}\right) + \frac{d_1 d_2}{f_1 f_2}\left(\frac{1}{f_3}+\frac{1}{f_4}\right) + \frac{d_2 d_3}{f_3 f_4}\left(\frac{1}{f_1}+\frac{1}{f_2}\right) \\ & + \frac{d_1 d_3}{f_1 f_4}\left(\frac{1}{f_2}+\frac{1}{f_3}\right) - \frac{d_1 d_2 d_3}{f_1 f_2 f_3 f_4}\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{L}{f} = {} & (d_1+d_2+d_3+d_4)\left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}\right) - \frac{d_1 d_2}{f_2}\left(\frac{1}{f_3}+\frac{1}{f_4}+\frac{1}{f_1}\right) \\ & - \frac{d_2 d_3}{f_3}\left(\frac{1}{f_4}+\frac{1}{f_1}+\frac{1}{f_2}\right) - \frac{d_3 d_4}{f_4}\left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}\right) - \frac{d_4 d_1}{f_1}\left(\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}\right) \\ & - d_1 d_3\left(\frac{1}{f_1}+\frac{1}{f_4}\right)\left(\frac{1}{f_2}+\frac{1}{f_3}\right) - d_2 d_4\left(\frac{1}{f_1}+\frac{1}{f_2}\right)\left(\frac{1}{f_3}+\frac{1}{f_4}\right) \\ & + \frac{d_1 d_2 d_3}{f_2 f_3}\left(\frac{1}{f_4}+\frac{1}{f_1}\right) + \frac{d_2 d_3 d_4}{f_3 f_4}\left(\frac{1}{f_1}+\frac{1}{f_2}\right) + \frac{d_3 d_4 d_1}{f_4 f_1}\left(\frac{1}{f_2}+\frac{1}{f_3}\right) \\ & + \frac{d_4 d_1 d_2}{f_1 f_2}\left(\frac{1}{f_3}+\frac{1}{f_4}\right) - \frac{d_1 d_2 d_3 d_4}{f_1 f_2 f_3 f_4}\end{aligned} \tag{8}$$

$$L_4 = f_{1234}(L/F) = H_1 + H_2 + d_4 \tag{9}$$

$$\begin{aligned}H_1 = f_{1234}\Bigg[& \frac{d_3}{f_4} + d_2\left(\frac{1}{f_3}+\frac{1}{f_4}\right) + d_1\left(\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}\right) - \frac{d_2 d_3}{f_3 f_4} \\ & - \frac{d_3 d_1}{f_4}\left(\frac{1}{f_2}+\frac{1}{f_3}\right) - \frac{d_1 d_2}{f_2}\left(\frac{1}{f_3}+\frac{1}{f_4}\right) + \frac{d_1 d_2 d_3}{f_2 f_3 f_4}\Bigg]\end{aligned} \tag{10}$$

$$\begin{aligned}H_2 = f_{1234}\Bigg[& \frac{d_1}{f_1} + d_2\left(\frac{1}{f_1}+\frac{1}{f_2}\right) + d_3\left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}\right) - \frac{d_1 d_2}{f_1 f_2} \\ & - \frac{d_1 d_3}{f_1}\left(\frac{1}{f_2}+\frac{1}{f_3}\right) - \frac{d_2 d_3}{f_3}\left(\frac{1}{f_1}+\frac{1}{f_2}\right) + \frac{d_1 d_2 d_3}{f_1 f_2 f_3}\Bigg]\end{aligned} \tag{11}$$

$$S_4 = \tfrac{1}{2}L_4 - H_2\,. \tag{12}$$

Less general ring resonators can be reduced to an iterated sequence of three, two, or one lens, respectively. The corresponding expressions for an iterated sequence of three thin lenses can be obtained from those for

four lenses by putting $d_4 = 1/f_4 = 0$ with appropriate redefinition of H_2 to locate the right-hand principal plane relative to lens f_3 :

$$\frac{1}{f_{123}} = \left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) - \frac{d_1}{f_1}\left(\frac{1}{f_2} + \frac{1}{f_3}\right) - \frac{d_2}{f_3}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) + \frac{d_1 d_2}{f_1 f_2 f_3} \quad (13)$$

$$\begin{aligned} \frac{L}{f} = (d_1 + d_2 + d_3)\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) - \frac{d_1 d_2}{f_2}\left(\frac{1}{f_3} + \frac{1}{f_1}\right) \\ - \frac{d_2 d_3}{f_3}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d_3 d_1}{f_1}\left(\frac{1}{f_2} + \frac{1}{f_3}\right) + \frac{d_1 d_2 d_3}{f_1 f_2 f_3} \end{aligned} \quad (14)$$

$$L_3 = f_{123}(L/f) = d_3 + H_1 + H_2 \quad (15)$$

$$H_1 = f_{123}\left[\frac{d_2}{f_3} + d_1\left(\frac{1}{f_2} + \frac{1}{f_3}\right) - \frac{d_1 d_2}{f_2 f_3}\right] \quad (16)$$

$$H_2 = f_{123}\left[\frac{d_1}{f_1} + d_2\left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d_1 d_2}{f_1 f_2}\right] \quad (17)$$

$$S_3 = \tfrac{1}{2}L_3 - H_2 = f_{123}\left[\frac{1}{2}\frac{L}{f} - \frac{d_1}{f_1} - d_2\left(\frac{1}{f_1} + \frac{1}{f_2}\right) + \frac{d_1 d_2}{f_1 f_2}\right]. \quad (18)$$

Similarly, for an iterated sequence of two thin lenses, f_1 , d_1 , f_2 , d_2 we obtain:

$$\frac{1}{f_{12}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d_1}{f_1 f_2} \quad (19)$$

$$\frac{L}{f} = (d_1 + d_2)\left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d_1 d_2}{f_1 f_2} \quad (20)$$

$$L_2 = f_{12}(L/f) = d_2 + h_1 + h_2 \quad (21)$$

$$h_1 = \frac{f_{12}\, d_1}{f_2} \quad (22)$$

$$h_2 = \frac{f_{12}\, d_1}{f_1} \quad (23)$$

$$S_2 = \tfrac{1}{2}L_2 - h_2 = f_{12}\left[\frac{1}{2}\frac{L}{f} - \frac{d_1}{f_1}\right]. \quad (24)$$

III. MATRIX REPRESENTATION OF LENS GROUP

A second method of analyzing an infinite sequence of thin-lens groups has been derived recently by Kogelnik,[13] based on the representation of a lens system by a 2-by-2 matrix of transmission-line parameters A, B,

C, D. He has shown that the same $ABCD$ matrix which describes the transformation of position and slope of a ray, between input and output planes of the system, also serves to transform the radius w and wavefront curvature R of a Gaussian beam. The transformation is expressed by

$$q_j = (Aq_i + B)/(Cq_i + D) \tag{25}$$

where i refers to the input plane and j to the exit plane of the system, and

$$\frac{1}{q} = \frac{1}{R} - j\frac{\lambda}{\pi w^2}. \tag{26}$$

The ray matrix for any lens system is given by

$$[a] = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1 - \frac{h_2}{f} & h_1 + h_2 - \frac{h_1 h_2}{f} \\ -\frac{1}{f} & 1 - \frac{h_1}{f} \end{vmatrix} \tag{27}$$

where h_1 and h_2 locate the principal planes relative to the input and output planes, respectively (Fig. 3), and f is the focal length of the system. Because of reciprocity,

$$AD - BC = 1. \tag{28}$$

For an element consisting of a thin lens f_1 followed by a distance d_1, we have

$$f = f_1, \qquad h_1 = 0, \qquad h_2 = d_1. \tag{29}$$

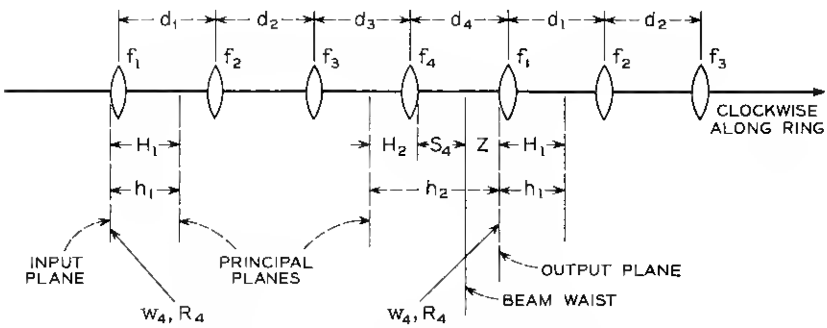

Fig. 3 — Optical system of four thin lenses for transformation of a beam into itself at the reference planes.

The ray matrix of this element is thus given by

$$[a_1] = \begin{vmatrix} 1 - \dfrac{d_1}{f_1} & d_1 \\ -\dfrac{1}{f_1} & 1 \end{vmatrix}. \tag{30}$$

The transformation of a beam into itself, after traversing a group of (say) four thin lenses, each followed by a spacing as indicated in Fig. 3, is then expressed by evaluating tbe $ABCD$ parameters of tbe product matrix:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = [a_4] \times [a_3] \times [a_2] \times [a_1] \tag{31}$$

wherein each element matrix has the same form as (30); and then setting $q_j = q_i$ in (25), to obtain (at tbe reference plane):

$$R_4 = \frac{2B}{D - A} \tag{32}$$

$$\frac{\pi w_4^2}{\lambda} = \frac{2B}{\sqrt{4 - (A + D)^2}} \tag{33}$$

Combining these expressions with relations (3)–(6), we find the spot size w_0 of the beam waist to be given by

$$\pi w_0^2/\lambda = -\sqrt{4 - (A + D)^2}/2C \tag{34}$$

and the location of that waist given by

$$z = (A - D)/2C \tag{35}$$

where z is measured from the waist to the reference plane (just in front of f_1 in Fig. 3).

The expressions for the equivalent thick lens of Fig. 2, evaluated in the previous section, can be related to the $ABCD$ parameters of the lens system of Fig. 3 with the belp of (27), as follows:

$$H_1 = h_1 = (D - 1)/C \tag{36}$$

$$H_2 + d_4 = h_2 = (A - 1)/C \tag{37}$$

$$L_4 = h_2 + h_1 = (A + D - 2)/C \tag{38}$$

$$\frac{1}{f_{1234}} = -C \tag{39}$$

$$\frac{L}{f} = 2 - (A + D) \tag{40}$$

$$z = d_4 - S_4 = \tfrac{1}{2}L_4 - H_1 \tag{41}$$

where z is also given by (35).

The $ABCD$ parameters for the four-lens system of Fig. 3 have been evaluated as indicated in (30) and (31) and listed in the Appendix. The parameters for a similar three-lens group f_1, d_1, $\cdots$ f_3, d_3 can be found by setting $d_4 = f_4^{-1} = 0$; and for a two-lens group by setting in addition $d_3 = f_3^{-1} = 0$.

Although the ray matrix formalism of Kogelnik offers no economy in computational labor over the straightforward derivation of the equivalent thick lens parameters, it has greater analytical flexibility and permits almost automatic extension to any number of lenses.

IV. ACKNOWLEDGMENT

The writer wishes to thank H. Kogelnik for helpful advice and stimulating discussions.

APPENDIX

Ray Matrix Parameters of Four-Lens System (Fig. 3)

$$\begin{aligned}
A = 1 - \frac{d_1}{f_1} - d_2\left(\frac{1}{f_1} + \frac{1}{f_2}\right) - d_3\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) \\
- d_4\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4}\right) \\
+ \frac{d_1 d_2}{f_1 f_2} + \frac{d_2 d_3}{f_3}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) + \frac{d_3 d_1}{f_1}\left(\frac{1}{f_2} + \frac{1}{f_3}\right) \\
+ \frac{d_4 d_1}{f_1}\left(\frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4}\right) + \frac{d_3 d_4}{f_4}\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) \\
+ d_2 d_4\left(\frac{1}{f_1} + \frac{1}{f_2}\right)\left(\frac{1}{f_3} + \frac{1}{f_4}\right) \\
- \frac{d_1 d_2 d_3}{f_1 f_2 f_3} - \frac{d_2 d_3 d_4}{f_3 f_4}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d_3 d_4 d_1}{f_4 f_1}\left(\frac{1}{f_2} + \frac{1}{f_3}\right) \\
- \frac{d_4 d_1 d_2}{f_1 f_2}\left(\frac{1}{f_3} + \frac{1}{f_4}\right) + \frac{d_1 d_2 d_3 d_4}{f_1 f_2 f_3 f_4}
\end{aligned} \tag{42}$$

$$
\begin{aligned}
B = {} & (d_1 + d_2 + d_3 + d_4) - \frac{d_1}{f_2}(d_2 + d_3 + d_4) - \frac{d_4}{f_4}(d_1 + d_2 + d_3) \\
& - \frac{(d_1 + d_2)(d_3 + d_4)}{f_3} + \frac{d_1 d_2}{f_2 f_3}(d_3 + d_4) + \frac{d_3 d_4}{f_3 f_4}(d_1 + d_2) \qquad (43) \\
& + \frac{d_4 d_1}{f_4 f_2}(d_2 + d_3) - \frac{d_1 d_2 d_3 d_4}{f_2 f_3 f_4}
\end{aligned}
$$

$$
\begin{aligned}
C = {} & -\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4}\right) + \frac{d_1}{f_1}\left(\frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4}\right) \\
& \qquad + d_2\left(\frac{1}{f_1} + \frac{1}{f_2}\right)\left(\frac{1}{f_3} + \frac{1}{f_4}\right) \\
& + \frac{d_3}{f_4}\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) - \frac{d_1 d_2}{f_1 f_2}\left(\frac{1}{f_3} + \frac{1}{f_4}\right) - \frac{d_2 d_3}{f_3 f_4}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) \qquad (44) \\
& - \frac{d_1 d_3}{f_4 f_1}\left(\frac{1}{f_2} + \frac{1}{f_3}\right) + \frac{d_1 d_2 d_3}{f_1 f_2 f_3 f_4}
\end{aligned}
$$

$$
\begin{aligned}
D = {} & 1 - d_1\left(\frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4}\right) - d_2\left(\frac{1}{f_3} + \frac{1}{f_4}\right) - \frac{d_3}{f_4} \\
& + \frac{d_1 d_2}{f_2}\left(\frac{1}{f_3} + \frac{1}{f_4}\right) + \frac{d_1 d_3}{f_4}\left(\frac{1}{f_2} + \frac{1}{f_3}\right) + \frac{d_2 d_3}{f_3 f_4} - \frac{d_1 d_2 d_3}{f_2 f_3 f_4}. \qquad (45)
\end{aligned}
$$

REFERENCES

1. Rosenthal, A. H., J. Opt. Soc. Am. *52*, Oct., 1962, p. 1143.
2. Macek, W. M., and Davis, D. T. M., Appl. Phys. Lett. *2*, Feb., 1963, p. 67.
3. Macek, W. M., Schneider, J. R., and Salamon, R. M., J. Appl. Phys. *35*, Aug., 1964, p. 2556.
4. Clark, P. O., Proc. IEEE *51*, June, 1963, p. 949.
5. Boyd, G. D., and Gordon, J. P., B.S.T.J. *40*, March, 1961, p. 489.
6. Fox, A. G., and Li, T., B.S.T.J. *40*, March, 1961, p. 453.
7. Boyd, G. D., and Kogelnik, H., B.S.T.J. *41*, July, 1962, p. 1347.
8. Kogelnik, H., review article in *Advances in Lasers*, Dekker Pub., New York, 1964.
9. Collins, S. A., Appl. Opt., *3*, 1964, p. 1263.
10. Collins, S. A., and Davis, D. T. M., Appl. Opt. *3*, 1964, p. 1314.
11. Jenkins, F. A., and White, H. E., *Fundamentals of Optics*, 3rd ed., McGraw-Hill Book Co., New York, 1957, p. 94.
12. Pierce, J. R., *Theory and Design of Electron Beams*, D. Van Nostrand and Co., New York, 1954.
13. Kogelnik, H., Imaging of Optical Modes—Resonators with Internal Lenses, B. S. T. J., *44*, March, 1965, p. 455.